Assignment 8

Deadline: March 23, 2018

Hand in: Supplementary Problems no 1, 2, 4, 5.

Supplementary Problems

These problems are largely taken from Text. I have rearranged them.

1. Find the pointwise limit of $\{(\cos \pi x)^{2n}\}$, $x \in (-\infty, \infty)$, and show that it is uniformly convergent on $[a, b]$ as long as $[a, b] \cap \mathbb{Z}$ is empty.

2. Find the pointwise limit of $\{\text{Arctan } nx\}$ and show that this sequence of functions is uniformly convergent on $[a, \infty)$ for every positive a but not uniformly convergent on $(0, \infty)$. The function Arctan is the inverse function of the tangent function on $(-\infty, \infty)$ to $(-\pi/2, \pi/2)$. A sketch of their graphs will help.

3. Let $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. Prove that $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$ for $\alpha, \beta \in \mathbb{R}$.

4. Let the two sequences of functions $\{f_n\}$ and $\{g_n\}$ uniformly converge to f and g respectively in E.
   (a) Show that their product $\{f_n g_n\}$ converges to fg uniformly on E under the assumption that $\|f_n\| \leq M$, $\|g_n\| \leq N$ for all $n \geq 1$ for some M, N.
   (b) Let $f_n(x) = x + 1/n \rightrightarrows f(x) = x$ on $(-\infty, \infty)$. Show that $\{f_n^2\}$ does not converge uniformly to f^2. It shows that the assumption in (a) cannot be dropped.

5. Let $f_n \rightrightarrows f$ on $[a, b]$ and $\|f_n\|_\infty \leq M$. For every continuous function Φ on $[-M, M]$, show that $\Phi \circ f_n \rightrightarrows \Phi \circ f$ on $[a, b]$.

6. Show that the following sequences are not uniformly convergent:
   (a) $\{(\cos \pi x)^{2n}\}$ on $[a, b]$ where $[a, b] \cap \mathbb{Z} \neq \phi$.
   (b) $\{(x - 2)^{1/n}\}$ on $[2, 5]$
   (c) $\left\{\tan\left(\dfrac{n\pi x}{1 + 2n}\right)\right\}$ on $(0, 1)$.

7. (Optional) Let $f_n \in C[a, b]$ converge pointwisely to f on $[a, b]$. Suppose that $f_n \rightrightarrows f$ on (a, b). Show that $f \in C[a, b]$ and $f_n \rightrightarrows f$ on $[a, b]$.

8. Let $\{x_k\}$ be an enumeration of all rational numbers in $[0, 1]$. Define $h_n(x)$ to be 1 at $x = x_1, \ldots, x_n$ and to be zero otherwise. Using this sequence to show that pointwise limit of integrable functions may not be integrable. (I have done this in class. Want you to do it again.)